MAY 2012 **P/ID 37457/PMAH**

Time : Three hours Maximum : 100 marks

PART A — (10 × 2 = 20 marks)

Answer ALL questions.

Each question carries 2 marks.

1. Define a normed linear space and given an example.
2. Define a topological space and give an example.
3. Give an example of a topological space which is not a T_1-space.
4. What is Bolzano-Weierstrass property?
5. What do you mean by components of a space? How many component does a connected space have?
6. Define Banach space and give an example.
7. Define an orthonormal set and give an example.
8. Define adjoint of an operator and self adjoint operator.
9. Define Banach algebra and a Banach subalgebra.
10. Define a topological divisor of zero and show that the set of all topological divisors of zero is a subset of the set of singular elements.

PART B — (5 × 6 = 30 marks)

Answer ALL questions.

Each question carries 6 marks.

11. (a) Let X and Y be metric spaces and f a mapping of X into Y. Prove that f is continuous if and only if $f^{-1}(G)$ is open in X whenever G is open in Y.

Or

(b) Establish that R^n is complete.

12. (a) Define homeomorphism and prove that a one-to-one continuous mapping of a compact space onto a Hausdorff space is a homeomorphism.

Or

(b) Prove that a metric space is compact if and only if it is complete and totally bounded.

13. (a) Prove that a subspace of the real line R is connected if and only if it is an interval and show that the real line R is connected.

Or

(b) If N is a normed linear space, prove that the closed unit sphere s^* in N^* is a compact Hausdorff space in the weak* topology.

14. (a) State and prove the open mapping theorem.

Or

(b) Prove that any closed convex subset C of a Hilbert space H contain a unique vector of smallest norm.

15. (a) Define spectrum of an element x in a Banach algebra A and show that it is non-empty.

Or

(b) If $1-xr$ is regular, prove that $1-rx$ is also regular.

PART C — $(5 \times 10 = 50$ marks)

Answer ALL questions.

Each question carries 10 marks.

16. (a) State and prove the Baire's theorem.

Or

(b) Give an example of a one-to-one continuous mapping of one topological space onto another which is not a homeomorphism. Show also that a topological space X is metrizable if and only if there exists a homeomorphism of X onto a subspace of some metric space Y.

17. (a) State and prove the Tychonoff's theorem.

Or

(b) State and prove the Tietze extension theorem.

18. (a) Let X be an arbitrary completely regular space. Prove that there exists a compact Hausdorff space $\beta(X)$ with the following properties :

    (i) X is a dense subspace of $\beta(X)$

    (ii) Every bounded continuous real function defined on X has a unique extension to a bounded continuous real function defined on $\beta(X)$.

Or

(b) State and prove the Hahn-Banach theorem with all necessary details.

19. (a) State and prove the closed graph theorem.

Or

(b) Let H be Hilbert space and f be an arbitrary functional in H^*. Prove that there exist a unique vector y in H such that $f(x) = (x, y)$ for every x in H.

20. (a) Establish the result $r(x) = \lim \|x^n\|^{\frac{1}{n}}$.

Or

(b) State and prove the Gelfand-Neumark theorem.

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