MAY 2015 **P/ID 37457/PMAH**

Time : Three hours Maximum : 100 marks

PART A — (10 × 2 = 20 marks)

Answer ALL questions.

1. State Cauchy's inequality.

2. Define a topological space and give an example.

3. Define compact space.

4. Define T_1 – space.

5. What do you mean by totally disconnected space?

6. What is continuous linear transformations?

7. Define bounded linear transformation.

8. What do you mean by normal and unitary operators?

9. Define Banach algebra.

10. State Gelfand-Neumark theorem.

PART B — (5 × 6 = 30 marks)

Answer ALL questions.

11. (a) If X is a complete metric space and Y is a subspace of X then prove that Y is complete iff it is closed.

Or

(b) If X is a topological space and A a subset of X then prove that $\overline{A} = A \cup D(A)$.

12. (a) State and prove Heine-Borel theorem.

Or

(b) Prove that every sequentially compact metric space is totally bounded.

13. (a) Prove that any continuous image of a connected space is connected.

Or

(b) State and prove open mapping theorem.

14. (a) If M is a closed linear subspace of a Hilbert space H, then prove that $H = M \oplus M^{\perp}$.

Or

(b) Prove that every non zero Hilbert space contains a complete orthonormal set.

15. (a) Prove that Z is a subset of S.

Or

(b) If I is a proper closed two-sided ideal in A, then prove that A/I is a Banach algebra.

PART C — (5 × 10 = 50 marks)

Answer ALL questions.

16. (a) State and prove Cantor's intersection theorem.

Or

(b) If f and g are continuous real or complex functions defined on a topological. X then prove that $f+g$, αf and fg are also continuous. Furthermore, if f and g are real then prove that $f \wedge g$ and $f \vee g$ are continuous.

17. (a) State and prove Tychonoff's theorem.

Or

(b) State and prove Urysohn's Lemma.

18. (a) Let X be a compact Hausdroff space. Then prove that X is totally disconnected iff it has an open base whose sets are also closed.

Or

(b) If N and N' are normed linear spaces, then prove that the set $\mathcal{B}(N, N')$ of all continuous linear transformations of N into N' is itself a normed linear space with respect to the pointwise linear operations.

19. (a) Prove that A closed convex subset C of a Hilbert space H contains a unique vector of smallest norm.

Or

(b) State and prove Bessel's inequality.

20. (a) Prove that the mapping $x \to x^{-1}$ of G into G is continuous and is therefore a homeomorphism of G onto itself.

Or

(b) Prove that the maximal ideal space $\mathcal{M}$ is a compact Hausdorff space.

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