(6 pages)

MAY 2014 **P/ID 37457/PMAH**

Time : Three hours Maximum : 100 marks

SECTION A — ($10 \times 2 = 20$ marks)

Answer ALL the questions.

1. Define a complete metric space.

2. Define subalgebra.

3. Define sequentially compact metric space.

4. Define Hausdorff space.

5. Define component of a space.

6. Prove that norm is a continuous function on a linear space.

7. Define Hilbert space.

8. Define an orthonormal set.

9. Define topological divisors of zero.

10. Define Banach* algebra.

SECTION B — ($5 \times 6 = 30$ marks)

Answer ALL the questions.

11. (a) Let X and Y be metric spaces and f be a mapping of X into Y. Prove that f is continuous $\Leftrightarrow f^{-1}(G)$ is open in X whenever G is open in Y.

Or

(b) State and prove Lindelof's theorem.

12. (a) Prove that every sequentially compact metric space is compact.

Or

(b) Prove that a one-to-one continuous mapping of a compact metric space onto a Hausdorff space is a homeomorphism.

13. (a) Prove that any continuous image of a connected space is connected.

Or

(b) State and prove Holder's inequality.

14. (a) If M and N are closed linear subspaces of a Hilbert space H such that $M \perp N$, prove that the linear subspace $M + N$ is also closed.

Or

(b) Prove that $y \to f_y$ is a norm–preserving mapping of H into H^*.

15. (a) If A is a Banach subalgebra of a Banach algebra A'. Then prove that the spectra of an element x in A with respect to A and A' are related as follows:

(i) $\sigma_{A'}(x) \subseteq \sigma_A(x)$;

(ii) Each boundary point of $\sigma_A(x)$ is also a boundary point of $\sigma_{A'}(x)$.

Or

(b) With usual rotation if r is an element of A with the property that $1 - x\,r$ is regular for every x, prove that r is in R.

SECTION C — ($5 \times 10 = 50$ marks)

Answer ALL the questions.

16. (a) State and prove Cauchy's inequality.

Or

(b) (i) Let X be a topological space and A be an arbitrary subset of X. Prove that $\overline{A} = \{x :$ each neighbourhood of x intersects $A\}$

(ii) Let X be a topological space and A be a subset of X. Prove that $\overline{A} = A \cup D(A)$; and A is closed $\Leftrightarrow A \supseteq D(A)$.

17. (a) Prove that a topological space is compact if every subbasic open cover has a finite subcover.

Or

(b) State and prove Ascoli's theorem.

18. (a) State and prove Wierstrass approximation theorem.

Or

(b) If N and N' are normed linear spaces, prove that the set of all continuous linear transformations of N into N' is itself a normed linear space with respect to the pointwise linear operations and the norm defined by $\|T\| = \sup\{\|T(x)\| : \|x\| \leq 1\}$.

Further if N' is a Banach space, prove that $B(N, N')$ is also a Banach space.

19. (a) State and prove the open mapping theorem. (Also prove the lemma).

Or

(b) Let H be a Hilbert space and let $\{e_i\}$ be an orthonormal set in H. Prove that the following conditions are equivalent:

(i) $\{e_i\}$ is complete;

(ii) $x \perp \{e_i\} \Rightarrow x = 0$;

(iii) if x is an arbitrary vector in H. then $x = \Sigma(x, e_i)e_i$;

(iv) if x is an arbitrary vector in H, then $\|x\|^2 = \Sigma|(x, e_i)|^2$;

20. (a) Prove that the mapping $x \to x^{-1}$ of G into G is continuous and is a homeomorphism as of G into itself.

Or

(b) State and prove Gelfand-Neumark representation theorem.