(6 pages)

MAY 2011 **P/ID 37457/PMAH**

Time : Three hours Maximum : 100 marks

SECTION A — (10 × 2 = 20 marks)

Answer ALL questions.

Each question carries 2 marks.

1. Define a Banach algebra.

2. Define an algebra.

3. State Tychonoff's theorem.

4. Define Hausdorff space.

5. Define totally disconnected space.

6. State Hahn Banach theorem.

7. State Open mapping theorem.

8. State Pythagorean theorem.

9. Define the spectrum of T.

10. Define Banach *– algebra.

SECTION B — (5 × 6 = 30 marks)

Answer ALL questions.

Each question carries 6 marks.

11. (a) Let X and Y be metric spaces and f be a mapping of X into Y. Prove that f is continuous if and only if $x_n \to x \Rightarrow f(x_n) \to f(x)$.

Or

(b) Let X be a topological space and A an arbitrary subset of X. Prove that $\overline{A} = \{x : \text{each neighbourhood of } x \text{ intersects } A\}$.

12. (a) Prove that a topological space is compact if and only if every basic open cover has a finite subcover.

Or

(b) Prove that any continuous mapping of a compact metric space into a metric space is uniformly continuous.

13. (a) Prove that any continuous image of a connected space is connected.

Or

(b) State and prove Holder's inequality.

14. (a) State and prove Schwarz inequality.

Or

(b) If T is an operator on H, prove that T is normal $\Leftrightarrow$ its real and imaginary parts commute.

15. (a) If $1-xr$ is regular, prove that $1-rx$ is also regular.

Or

(b) If f_1 and f_2 are functionals on a Banach algebra A with the same null space M, prove that $f_1 = f_2$.

SECTION C — (5 × 10 = 50 marks)

Answer ALL questions.

Each question carries 10 marks.

16. (a) Let X be a metric space, let Y be a complete metric space, and let A be a dense subspace of X. If f is a uniformly continuous mapping of A into Y, prove that f can be extended uniquely to a uniformly continuous mapping g of X into Y.

Or

(b) (i) Prove that every separable metric space is second countable.

(ii) Let X be a topological space and let $\{f_n\}$ be a sequence of real or complex functions defined on X which converges uniformly to a function f defined on X. If all the f_n's are continuous, prove that f is also continuous.

17. (a) State and prove Ascoli's theorem.

Or

(b) State and prove Urysohn Imbedding theorem.

18. (a) Let M be a closed linear subspace of a normed linear space N. If the norm of a cos et $x + M$ in the quotient space N/M is defined by

$$\|x + M\| = \inf\{\|x + m\| : m \in M\},$$

then prove that N/M is a normed linear space. Further, if N is a Banach space, then prove that N/M is also a Banach space.

Or

(b) If N is a normed linear space, prove that the closed unit sphere S* in N* is a compact Hausdorff space in the weak* topology.

19. (a) State and prove the closed graph theorem.

Or

(b) Let H be a Hilbert space and let f be an arbitrary functional in H*. Prove that there exists a unique vector y in H such that $f(x) = (x, y)$ for every x in H.

20. (a) Prove that the mapping $x \to x^{-1}$ of G into G is continuous and is therefore a homeomorphism of G onto itself.

Or

(b) Prove that $r(x) = \lim \|x\|^{1/n}$.