(6 pages)

OCTOBER 2012 **P/ID 4525/XDB**

Time : Three hours Maximum : 100 marks

SECTION A — (4 × 20 = 80 marks)

Answer ALL the questions.

Each question carries 20 marks.

1. (a) (i) State and prove the Cantor intersection theorem.

(ii) Prove that the following three statements are equivalent for any subset S of $\mathbf{R}^n$.

(1) S is compact.

(2) S is closed and bounded.

(3) Every infinite subset of S has an accumulation point in S.

(6 + 14 = 20)

Or

(b) (i) Let $f : S \to T$ be a function from one metric space (S, d_S) to another (T, d_T). Let A be a compact subset of S. Prove that if f is continuous on A then f is uniformly continuous on A.

(ii) Assume that α is of bounded variation on $[a,b]$. Let $V(x)$ denote the total variation of α on $[a,x]$ if $a < x \leq b$ and let $V(a) = 0$. Let f be defined and bounded on $[a,b]$. If $f \in R(\alpha)$ on $[a,b]$ then prove that $f \in R(V)$ on $[a,b]$.

(10 + 10 = 20)

2. (a) (i) Assume that $\lim_{p,q \to \infty} f(p,q) = a$. For each fixed p, assume that the limit $\lim_{q \to \infty} f(p,q)$ exists. Then prove that the limit $\lim_{p \to \infty} (\lim_{q \to \infty} f(p,q))$ also exists and has the value a.

(ii) State and prove Merten's theorem.

(5 + 15 = 20)

Or

(b) (i) State and prove Bernstein's theorem.

(ii) State and prove Tauber's first theorem.

(10 + 10 = 20)

3. (a) (i) State and prove the chain rule for differentiation of vector-valued functions.

(ii) If both partial derivatives $D_r\vec{f}$ and $D_k\vec{f}$ exists in an n-ball $B(\vec{c})$ and if both $D_{r,k}\vec{f}$ and $D_{k,r}\vec{f}$ are continuous at $\vec{c}$, then prove that $D_{r,k}\vec{f}(\vec{c}) = D_{k,r}\vec{f}(\vec{c})$.

(5 + 15 = 20)

Or

(b) (i) Let $f \in L^2[-\pi, \pi]$ and let T_n be any trigonometric polynomial of degree n. Then prove that $\| f - T_n\|_2 \geq \| f - s_n\|_2$ where

$$s_n(t) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt)$$

$(-\pi \leq t \leq \pi)$ and the a_k, b_k are the Fourier coefficients of f.

(ii) Derive Bessel's inequality for generalised Fourier series. (14 + 6 = 20)

4. (a) (i) If G_1 and G_2 are open subsets of $[a,b]$ then prove that

$$|G_1| + |G_2| = |G_1 \cup G_2| + |G_1 \cap G_2|.$$

(ii) If f is a bounded measurable function on $[a,b]$ then prove that $f \in L[a,b]$.

(10 + 10 = 20)

Or

(b) (i) State and prove Lebesgue dominated convergence theorem.

(ii) Prove that the metric space $L^2[a,b]$ is complete. (8 + 12 = 20)

SECTION B — (10 × 2 = 20 marks)

Answer any TEN questions.

Each question carries 2 marks.

5. Define a bounded subset of $\mathbf{R}^n$.

6. Give a countable covering of $(0,1)$.

7. Justify the following statement :

Every curve in $\mathbf{R}^n$ is connected.

8. If α is increasing on $[a,b]$ and if $f \in R(\alpha)$ on $[a,b]$, prove that $f^2 \in R(\alpha)$ on $[a,b]$.

9. If $f(p,q) = \dfrac{pq}{p^2 + q^2}$, $(p = 1, 2, ..., q = 1, 2, ...)$.

   Find $\lim_{p\to\infty} (\lim_{q\to\infty} f(p,q))$.

10. Show that the series $\sum(-1)^{n+1} n$ is not $(c,1)$ summable.

11. Define a uniformly convergent sequence of functions.

12. State Abel's limit theorem.

13. Prove that if $\vec{f}$ is differentiable at $\vec{c}$, then $\vec{f}$ is continuous at $\vec{c}$.

14. Let $\vec{f} : \mathbf{R}^n \to \mathbf{R}$ be defined by $\vec{f}(\vec{x}) = \vec{a} . \vec{x}$ where $\vec{a}$ is a fixed vector in $\mathbf{R}^n$. Calculate all first order partial derivatives of $\vec{f}$.

15. State true or false. Justify your assertion. If f and g are two functions in $L^2[-\pi, \pi]$ that have the same Fourier coefficients, then $f = g$.

16. If G is an open subset of $[a,b]$, define the length of G.

17. If $mE = 0$ then show that every subset of E is measurable and has measure zero.

18. Prove that the characteristic function of the rational numbers in [0,1] is a measurable function.

19. If E is a measurable subset of $[a,b]$, evaluate $\int_E 1$.