(6 pages)

DECEMBER 2014 **P/ID 37452/PMAB**

Time : Three hours Maximum : 100 marks

PART A — (10 × 2 = 20 marks)

Answer ALL questions.

1. If 'f' is monotonic on $[a, b]$, then prove that 'f' is of bounded variation on $[a, b]$.

2. If $f(x) = 0$ for all irrational x and $f(x) = 1$ for all rational x, then prove that $f \notin R(\alpha)$ on $[a, b]$ for any $a < b$.

3. State second fundamental theorem of integral calculus.

4. State Bernstein's theorem for convergence of Taylor's series.

5. Give an example of a sequence of continuous functions with a discontinuous limit function.

6. State Riemann-Lebesgue lemma.

7. Prove that constant functions are measurable.

8. Prove that if 'f' is integrable, then f is finite-valued almost everywhere.

9. State first-order Taylor formula.

10. State inverse function theorem.

PART B — ($5 \times 6 = 30$ marks)

Answer ALL questions.

11. (a) If α is increasing on $[a, b]$ and $f \in R(\alpha)$ on $[a, b]$, then prove that $|f| \in R(\alpha)$ on $[a, b]$ and $\left|\int_a^b f(x)\,d\alpha(x)\right| \leq \int_a^b |f(x)|\,d\alpha(x)$.

Or

(b) If a series is convergent with sum S, then prove that it is also (C, 1) summable with Cesaro sum S.

12. (a) State and prove first mean-value theorem for Riemann Stieltjes integral.

Or

(b) State and prove Tauber's theorem.

13. (a) State and prove Dirichlet's test for uniform convergence of a series.

Or

(b) State and prove Riemann's localization theorem of a fourier series.

14. (a) Let f and g be real-valued measurable functions defined on the same measurable set E. Then prove that $f+g$, $f-g$ and $f \cdot g$ are also measurable.

Or

(b) Let f be bounded and measurable on a finite interval $[a, b]$ and let $\in > 0$. Then prove that there exist.

(i) a step function 'h' such that $\int_a^b |f-h| dx < \in$,

(ii) a continuous function 'g' such that 'g' vanishes outside a finite interval and $\int_a^b |f-g| dx < \in$.

15. (a) State and prove chain rule for the differentiable functions of several variable.

Or

(b) Let $f = (f_1, f_2, ..., f_n)$ have continuous partial derivatives $D_i f_i$ on an open set S in R^n, and the Jacobian determinant $J_f(a) \neq 0$ for some point 'a' in S. Then prove that there is an n-ball $B(a)$ on which 'f' is one-to-one.

PART C — (5 × 10 = 50 marks)

Answer ALL questions.

16. (a) State and prove integration by parts formula for Riemann-Stieltjes integral.

Or

(b) State and prove Riemann's theorem on conditionally convergent series.

17. (a) State and prove rearrangement theorem for double series.

Or

(b) State and prove the theorem on change of variable in Riemann integral.

18. (a) Let α be of bounded variation on $[a, b]$. Assume that each term of the sequence $\{f_n\}$ is a real-valued function such that $f_n \in R(\alpha)$ on $[a, b]$ for each $n = 1, 2, \ldots$. Let $f_n \to f$ uniformly on $[a, b]$ and define $g_n(x) = \int_a^x f_n(t)\, d\alpha(t)$ if $x \in [a, b]$, $n = 1, 2, \ldots$ then prove that

(i) $f \in R(\alpha)$ on $[a, b]$

(ii) $g_n \to g$ uniformly on $[a, b]$, where $g(x) = \int_a^x f(t)\, d\alpha(t)$.

Or

(b) State and prove Fejer's theorem and hence deduce Weierstrass approximation theorem.

19. (a) Prove that not every measurable set is a Borel set.

Or

(b) State and prove Fatou's lemma and hence deduce Lebesgue's monotone convergence theorem.

20. (a) State and prove implicit function theorem.

Or

(b) Assume that one of the partial derivatives $D_1 f_1 ,...., D_n f$ exists at 'C' and that the remaining $(n-1)$ partial derivatives exist in some n-ball $B(C)$ and are continuous at 'C'. Then prove that 'f' is differentiable at 'C'

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