(6 pages)

MAY 2012 **P/ID 37453/PMAC**

Time : Three hours Maximum : 100 marks

SECTION A — (10 × 2 = 20 marks)

Answer ALL questions.

Each question carries 2 marks.

1. Define the characteristic polynomial.

2. Define linearly independent solutions and give an example.

3. Define the Wronskian $W(\phi_1,....\phi_n)$.

4. Give a criterion for the linear independence of the n solutions $\phi_1,....\phi_n$ of $L(y)=0$ on an interval I in terms of Wronskian.

5. State a necessary and sufficient condition for a differential equation to be exact.

6. Verify that x^2 is a solution of $x^2 y'' + xy' - 4y = 0$.

7. Check whether the equation $\left(y^2\, e^{xy^2} + 4x^3\right) dx + \left(2xy\, e^{xy^2} - 3y^z\right) dz = 0$ is exact.

8. State Cauchy's problem.

9. Define initial polynomial.

10. Explain the concept of Boundary value problem.

SECTION B — ($5 \times 6 = 30$ marks)

Answer ALL questions.

Each question carries 6 marks.

11. (a) Let ϕ_1 and ϕ_2 be two solutions of $L(y)=0$ on an interval I and set x_0 be any point in I. Prove that ϕ_1, ϕ_2 are linearly independent on I if and only if $W(\phi_1, \phi_2)(x_0) \neq 0$.

Or

(b) Let ϕ_1, ϕ_2 be any two linearly independent solutions of $L(y)=0$ on an interval I. Prove that every solution ϕ of $L(y)=0$ can be uniquely written as $\phi = c_1\, \phi_1 + c_2 \phi_2$ where c_1 and c_2 are constants.

12. (a) Prove that there exist n linearly independent solutions of $L(y)=0$ on I.

Or

(b) If ϕ_1 is a solution of $L(y) = y'' + a_1(x)\, y' + a_2(x) y = 0$ on I, and $\phi_1(x) \neq 0$ on I, find a second solution $\phi_2\,(x)$ on I.

13. (a) Find a basis for the solutions of $x^2 y'' + xy' + y = 0$ for $x \neq 0$.

Or

(b) Let M, N be two real-valued functions which have continuous first partial derivatives on some rectangle.

$R : |x - x_0| \leq a, |y - y_0| \leq b$

If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ then prove that

$M(x, y) + N(x, y)\, y' = 0$ is exact.

14. (a) Reduce the equation into canonical form $(1 - x^2) u_{xx} - (1 - y^2)^2 u_{yy} = 0$.

Or

(b) Determine the D' Alembert's solution of the one-dimensional wave equation.

15. (a) With the usual notation prove that $g(y)\, \delta(y - b) = g(b)\, \delta(y - b)$.

Or

(b) State the maximum principle and prove it.

SECTION C — (5 × 10 = 50 marks)

Answer ALL questions.

Each question carries 10 marks.

16. (a) For any real number x_0 and constants α, β prove that there exists a solution ϕ of the initial value problem $L(y)=0, y(x_0)=\alpha$, $y'(x_0)=\beta$.

Or

(b) If ϕ_1, ϕ_2 are two solutions of $L(y)=0$ on an interval I containing a point x_0 then prove that

$$W(\phi_1, \phi_2)(x) = e^{-\alpha_1 (x-x_0)} \, w(\phi_1, \phi_2)(x_0).$$

17. (a) Let $\phi_1, \ldots \phi_n$ be the n solutions of $L(y)=0$ on I satisfying $\phi_i^{(i-1)}(x_0)=1, \phi_i^{(j-1)}(x_0)=0$, $j \neq i$. If ϕ is any solution of $L(y)=0$ on I, prove that there are n constants $c_1, \ldots c_n$ such that $\phi = c_1 \phi_1 + \ldots + c_n \phi_n$.

Or

(b) Let ϕ_1 be a solution of $L(y)=0$ on an interval I, and suppose $\phi_1(x)\neq 0$ on I. If $v_1,....v_n$ is any basis on I for the solutions of $\phi_1 v^{(n-1)}+...+\left[n\,\phi_1^{(n-1)}+(n-1)\,\phi_1^{(n-2)}+...\right.$ $\left.+\alpha_{n-1}\phi_1\right]\ v=0$, and if $v_k=u_k'\ (k=2,...n)$, then prove that $\phi_1,\ u_2\phi_1,....,u_n\,\phi_1$ is a basis for the solutions of $L(y)=0$ on I.

18. (a) Derive the solution of $x^2y''+xy^1+y=0$.

Or

(b) Let f be a continuous real-valued function on $R:|x-x_0|\leq a, |y-y_0|\leq b$, $(a,b\geq 0)$, and let $|f(x,y)|\leq m$ for all (x,y) in R. Further suppose that f satisfies a Lipschitz condition with constant K in R. Then prove that the successive approximations

$$\phi_0(x)=y_0$$

$$\phi_{k+1}(x)=y_0+\int_{x_0}^{x} f(t,\phi_k(t))\,dt \quad k=0,1,2,....$$

Converge on the interval $I=|x-x_0|\leq\alpha=\min\{a,b/M\}$ to ϕ.

19. (a) Discuss the solution of one-dimensional heat equation by separation of variables.

Or

(b) Prove the following :

$$\psi(r) = \frac{1}{r} e^{\pm ikr \pm i\,k\,ct}$$

$$\psi(r, \theta) = \frac{1}{r}\left[\frac{\sin(Kr)}{Kr} - \cos(Kr)\right] \cos\theta \, e^{\pm i\,k\,c\,t}$$ are particular solution of the wave equation $\Delta^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$.

20. (a) State and solve the Neumann problem.

Or

(b) State and solve the Dirichlet problem.