(7 pages)

OCTOBER 2013 **P/ID 37453/PMAC**

Time : Three hours Maximum : 100 marks

PART A — (10 × 2 = 20 marks)

Answer ALL questions.

Each question carries 2 marks.

1. Solve $y'' - 4y' + 5y = 0$.

2. Find the general solution of $y''' - 3y' - 2y = 0$.

3. Prove that $P_0(x) = 1$.

4. Verify whether the equation $e^x dx + e^y (y+1) dy = 0$ is exact.

5. Write down the one dimensional heat equation.

6. Write down the wave equation in cylindrical coordinates.

7. Define linear dependence and independence.

8. Find the adjoint operator of $L(u) = u_{xx} - u_t$.

9. Define Newman boundary value problem.

10. Define Green's function.

PART B — ($5 \times 6 = 30$ marks)

Answer ALL questions.

Each question carries 6 marks.

11. (a) Let ϕ_1, ϕ_2 be two solutions of $L(y)=0$ on an interval I and let x_0 be any point in I. Then prove that ϕ_1, ϕ_2 are linearly independent on I iff $w(\phi_1,\phi_2)(x_0)\neq 0$.

Or

(b) Let $\alpha_1,\ldots,\alpha_n$ be any n constants and let x_0 be any real number. On any interval I containing x_0 then prove that there exist at most one solution ϕ of $L(y)=0$ satisfying

$$\phi(x_0)=\alpha_1\ ,\ \phi'(x_0)=\alpha_2,\ldots,\alpha^{(n-1)}(x_0)=\alpha_n.$$

12. (a) Let x_0 be in I and let $\alpha_1,..,\alpha_n$ be any n constants. Prove that there is at most one solution ϕ of $L(y)=0$ on I satisfying

$$\phi(x_0)=\alpha_1\ ,\ \phi'(x_0)=\alpha_2,\ldots,\alpha^{(n-1)}(x_0)=\alpha_n.$$

Or

(b) Show that $x^{\frac{1}{2}}J_{-\frac{1}{2}}x=\dfrac{\sqrt{2}}{\Gamma\left(\frac{1}{2}\right)}\cos x$.

13\. (a) Suppose S is either a rectangle $|x - x_0| \le a$; $|y - y_0| \le b$; $a, b > 0$ or a strip

$|x - x_0| \le a$; $|y| < \infty$; $a > 0$ and f is a real valued function defined on S such that $\frac{\partial f}{\partial y}$ exist, f is continuous on S and

$\left|\frac{\partial f}{\partial y}(x, y)\right| \le k$ for some $k > 0$.

Prove that f satisfies a Lipschitz condition on S with Lipschitz constant K.

Or

(b) Find the solution of the wave equation $U_{tt} = c^2 U_{xx}$ under the following conditions.

(i) $u(0, t) = u(2, t) = 0$

(ii) $u(x, 0) = \sin^3 \frac{\pi x}{2}$

(iii) $u_t(x, 0) = 0$.

14. (a) Solve the one-dimensional diffusion equation in the region $0 \le x \le \pi$; $t \ge 0$ subject to the conditions.

(i) T remains finite as $t \to \infty$

(ii) T=0 if $x = 0$ and π for all t

(iii) At t=0; $T = \begin{cases} x & 0 \le x \le \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \le x \le \pi \end{cases}$

Or

(b) A thin membrance of great extent is released from rest in the position $z = f(x, y)$. Determine the displacement at any subsequent time.

15. (a) Show that $G(r_1.r_2) = G(r_2.r_1)$.

Or

(b) Determine Green's function for the Helmholtz equation for the half space $z \ge 0$.

PART C — (5 × 10 = 50 marks)

Answer ALL questions.

Each question carries 10 marks.

16. (a) Let ϕ be any solution of $L(y) = y'' + a_1 y' + a_2 y = 0$ on an interval I containing a point x_0. Then prove that for all x in I.

$\|\phi(x_0)\| e^{-k|x-x_0|} \leq \|\phi(x)\| \leq \|\phi(x_0)\| e^{k|x-x_0|}$ where $\|\phi(x)\| = \left[|\phi(x)|^2 + |\phi'(x)|^2\right]^{\frac{1}{2}}$; $K = 1 + |a_1| + |a_2|$.

Or

(b) Let ϕ be any solution of $L(y) = y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = 0$ on an interval I containing a point x_0. Then prove that for all x in I.

$\|\phi(x_0)\| e^{-k|x-x_0|} \leq \|\phi(x)\| \leq \|\phi(x_0)\| e^{-k|x-x_0|}$ where $K = 1 + |a_1| + \ldots |a_n|$.

17. (a) Let $\phi_1, \ldots, \phi_n$ be n linearly independent solutions of $L(y) = 0$ on an interval I. If ϕ is any solution of $L(y) = 0$ on I, it can be represented in the form $\phi = c_1\phi_1 + \ldots + c_n\phi_n$

where $c_1,...,c_n$ are constants. Then prove that any set of n linearly independent solutions of $L(y)=0$ on I is a basis for the solutions of $L(y)=0$ in I.

Or

(b) One solution of $y'' - \frac{2}{x^2}y = 0$; $(0 < x < \infty)$ is $\phi_1(x) = x^2$. Find basis for the solutions on $0 < x < \infty$.

18. (a) Use Green's function technique to solve the Dirichlet's problem for a semi infinite space.

Or

(b) State and prove the existence theorem for the initial value problem $y' = f(x,y); y(x_0) = y_0$ on I.

19. (a) The ends A and B of a rod 10 cm in length are kept at temperature 0°C and 100°C until the steady state conditions prevails. Suddenly the temperature at the end A is increased to 20°C and the end B is decreased to 60°C. Find the temperature distribution in the rod at time t.

Or

(b) Derive the D'Alembert solution for the one dimensional wave equation.

20. (a) Explain Interior and Exterior Dirchlet problems for Laplace equation.

Or

(b) Solve the Dirichlet problem

$$\nabla^2 u = 0,\ 0 < x < 1,\ 0 < y < 1$$
$$u(x,o) = x(x-1),\ \cdot 0 \le x \le 1$$
$$u(x,1) = u(0,y) = u(1,y) = 0.$$

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