MAY 2016 **P/ID 37455/PMAF**

Time : Three hours Maximum : 100 marks

PART A — (10 × 2 = 20 marks)

Answer ALL questions.

1. Compute $\int\limits_{|z|=1} \frac{e^z}{z} dz$.

2. State *Cauchy's estimate*

3. Define a potential function.

4. State Schwartz theorem.

5. Show that the product $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\ldots$ converges to $\frac{1}{2}$.

6. Prove that every function which is meromorphic in the whole plane is quotient of two entire functions.

7. Define *Schwarz-Christoffel* formula.

8. Prove that an elliptic function without poles is a constant.

9. Define a *sheaf*.

10. Define homotopic of two curves.

PART B — (5 × 6 = 30 marks)

Answer ALL questions.

11. (a) State and prove *maximum principle*.

Or

(b) State and prove *Rouche's* theorem.

12. (a) Find the poles and residues of $\frac{1}{z^2+1}$.

Or

(b) If u_1 and u_2 are harmonic functions in a region Ω then prove that $\int_\gamma u_1 * du_2 - u_2 * du_1 = 0$.

13. (a) If $\sum_{n=1}^{\infty} \log(1+a_n)$ is convergent then prove that $\prod_{n=1}^{\infty}(1+a_n)$ is convergent with $1+a_n \neq 0$.

Or

(b) Prove that $\xi(s) = \frac{1}{2} s(1-s)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\xi(s)$ is entire and satisfies $\xi(s) = \xi(1-s)$.

14. (a) Suppose the boundary of a simply connected region Ω contain a line segment γ as one-sided free bounded arc, show that function $f(z)$ which maps Ω onto the unit disc can be extended to a function which is analytic and one to one $\Omega \cup \gamma$.

Or

(b) Prove that the sum of residues of an elliptic function is zero.

15. (a) With usual notations, prove that $\wp(2z) = \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2 - 2\wp(z)$.

Or

(b) Derive *Legendre's* relation.

PART C — (5 × 10 = 50 marks)

Answer ALL questions.

16. (a) Suppose that $\phi(\zeta)$ is continuous on the arc γ. Then prove that the function $F_n(z) = \int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^n} d\zeta$ is analytic in each regions determined by γ and its derivative is $F_n'(z) = nF_{n+1}(z)$.

Or

(b) State and prove *residue* theorem.

17. (a) Evaluate $\int_0^\infty \frac{x^2}{(x^2+a^2)^3} dx, (a \; real)$.

Or

(b) State and prove *Laurent's* theorem.

18. (a) State and prove Mittag-Lefler theorem.

Or

(b) Prove that a family F of continuous functions with values in a metric space S is normal in the region Ω of the complex plane if and only if

(i) F is equicontinuous on every compact set $E \subseteq \Omega$;

(ii) For any $z \in \Omega$ the values $f(z), f \in F$, lie in compact subject of S.

19. (a) State and prove *Riemann-mapping* theorem.

Or

(b) State and prove *Harnack's* principle.

20. (a) Show that $\wp(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right)$ where the sum ranges over all $\omega = n_1\omega_1 + n_2\omega_2$ except zero.

Or

(b) Show that the homotopy classes of closed curves from z_0 with respect to the region Ω, form a group and as an abstract group, does not depend on the point z_0.