(6 Pages)

OCTOBER 2012 **P/ID 37451/PMAA**

Time : Three hours Maximum : 100 marks

SECTION A — ($10 \times 2 = 20$ marks)

Answer ALL questions.

Each question carries 2 marks.

1. If G is a group and $a \in G$, define the normalizer of a in G.

2. Define a finitely generated R-module.

3. Test the validity of the following statement: Every linear transformation on a vector space over every field F has all its characteristic roots in F.

4. Define a nilpotent linear transformation.

5. Define an algebraic number.

6. Prove that trace (A+B) = trace (A)+trace (B) for any two matrices A and B.

7. What can you say about any finite extension of a field of characteristic zero?

8. Let K be a field and G be a group of automorphisms of K. Define the fixed field of G.

9. Prove that every abelian group is solvable.

10. State the theorem of Frobenius.

SECTION B — ($5 \times 6 = 30$ marks)

Answer ALL questions.

Each question carries 6 marks.

11. (a) Derive the class equation of any finite group G.

Or

(b) Prove that a group G is solvable if and only if $G^{(K)} = (e)$ for some integer K.

12. (a) If V is n-dimensional over F and if T$\in$A(V) has all its characteristic roots in F, then prove that T satisfies a polynomial of degree n over F.

Or

(b) By a direct matrix computation, show that $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ are not similar.

13. (a) If L is an algebraic extension of K and if K is an algebraic extension of F, then prove that L is an algebraic extension of F.

Or

(b) Prove that a linear transformation T on V is unitary if and only if it takes an orthonormal basis of V onto an orthonormal basis of V.

14. (a) State and prove the remainder theorem.

Or

(b) Let K be a field and G be the group of automorphisms of K. Prove that the fixed field of G is a subfield of K.

15. (a) Suppose that the field F has all n^th^ roots of unity (for some particular n) and suppose that $a \neq 0$ is in F. Let $x^n - a \in F(x)$ and let K be its splitting field over F. Prove that $K = F(u)$ where u is any root of $x^n - a$.

Or

(b) Let F be a finite field with q elements and suppose that $F \subset K$ where K is also a finite field. Then prove that K has q^n elements where $n = [K : F]$

SECTION C — $(5 \times 10 = 50$ marks)

Answer ALL questions.

Each question carries 10 marks.

16. (a) State and prove Sylow's theorem.

Or

(b) Let R be a Euclidean ring. Prove that any finitely generated R-Module is the direct sum of a finite number of cyclic submodules.

17. (a) Prove that two nilpotent linear transformations are similar if and only if they have the same invariants.

Or

(b) If T in $A_F(v)$ has as minimal polynomial $p(x) = q(x)^e$, where $q(x)$ is a monic, irreducible polynomial in F[x], then prove that there exists a basis of V over F in which the matrix of T is of the form.

$$\begin{pmatrix} C(q(x)^{e_1}) & & & & & \\ & C(q(x)^{e_2}) & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & C(q(x)^{e_r}) \end{pmatrix}$$

where $e = e_1 \geq e_2 \geq ....e_r$. Here $C(q(x)^{e_i})$ $(1 \leq i \leq r)$ is the companion matrix of $q(x)^{e_i}$.

18. (a) If L is a finite extension of K and if K is a finite extension of F, Prove that L is a finite extension of F. Also prove that $[L:F]=[L:K][K:F]$.

Or

(b) Prove that two real symmetric matrices are congruent if and only if they have the same rank and signature.

19. (a) Let $f(x) \in F[x]$ be a polynomial of degree $n \geq 1$. Prove that there is an extension E of F of degree at most n! in which $f(x)$ has n roots.

Or

(b) Let K be a normal extension of F and let H be a subgroup of $G(K,F)$. Let $K_H = \{x \in K \mid \sigma(x) = x \text{ for all } \sigma \in H\}$ be the fixed field of H. Then prove that

(i) $[K:K_H] = o(H)$

(ii) $H = G(K, K_H)$.

20. (a) If $p(x) \in F(x)$ is solvable by radicals over F, prove that the Galois group over F of $p(x)$ is a solvable group.

Or

(b) Prove that for every prime number p and every positive integer m there is a unique field having p^m elements.
