(6 pages)

OCTOBER 2012 **P/ID 4524/XDA**

Time : Three hours Maximum : 100 marks

SECTION A — (4 × 20 = 80 marks)

Answer ALL questions.

Each questions carries 20 marks.

1. (a) (i) If p is a prime number and $p / O(G)$, then prove that G has an element of order p.

(ii) If L is a finite extension of K and if K is a finite extension of F, then show that L is a finite extension of F and deduce that $[K : F] / [L : F]$.

Or

(b) (i) If G is S_3 and $A = <(12)>$ is G, find all the double cosets AxA of A in G.

(ii) State and prove the fundamental theorem of finitely generated modules over Euclidean rings.

2. (a) (i) Show that any two splitting fields of the same polynomial over a given filed F are isomorphic by an isomorphism leaving every element of F fixed.

(ii) If F is of characteristic 0 and if a, b are algebraic over F, then show that there exists an elements $c \in F(a,b)$ such that $F(a,b) = F(c)$.

Or

(b) (i) Let K be a normal extension of F and let H be a subgroup of $G(K,F)$; Let $K_H = \{x \in K : \sigma(x) = x$ for all $\sigma \in H\}$ be the fixed of H. Then prove that $[K : K_H] = 0(H)$ and $H = G(K, K_H)$.

(ii) Let K be the splitting field of $f(x)$ in $F(x)$ and let $p(x)$ be an irreducible factor of $f(x)$ in $F(x)$. If the roots of $p(x)$ are $\alpha_1, \alpha_2 .... \alpha_n$, then prove that for each i there exists an automorphism σ_i in $G(K,F)$ such that $\sigma_i(\alpha_1) = \alpha_i$.

3. (a) (i) Show that a polynomial $p(x) \in F(x)$ is solvable by radicals over F if and only if the Galois. Group over F of $p(x)$ is a solvable group.

(ii) IF $T \in A(V)$ has all its characteristic roots in F, show that there is a basis of V in which the matrix of T is triangular.

Or

(b) (i) Show that there exists a subspace W of V, invariant under the nilpotent linear transformation, such that $V = V_1 \oplus W$, where V_1 is the subspace of V spanned by $v_1 = v, v_2 = vT, v_{n_1} = vT^{n_1 - 1}$.

(ii) Show that the multiplicative group of non zero elements of a finite field is cyclic.

4. (a) (i) Show that the elements S and T in $A_F(V)$ are similar in $A_F(V)$ if and only if they have the same elementary divisors.

(ii) If F is the field of rational numbers, find all possible rational canonical forms and elementary divisors for the 6×6 matrices in F_6 having $(x-1)(x^2+1)^2$ as minimal polynomial.

Or

(b) (i) If A is a triangular matrix, prove that the entries on the diagonal of A are exactly all the characteristic roots of A.

(ii) Show that the linear transformation T on V is unitary if and only if it taken an orthonormal basis of V into an orthonormal basis of V.

SECTION B — (10 × 2 = 20 marks)

Answer any TEN questions.

Each questions carries 2 marks.

5. Let G be a finite group and $a \in G$ be such that a has only two conjugates. Prove that $N(a)$ is a normal subgroup of G.

6. Let G be a group and $a \in G$. Prove that $a \in Z(G)$ if only if $C(a) = \{a\}$.

7. Prove that a group of order 121 is abelian.

8. Find $\left[F\left(\sqrt{2}, \sqrt{3}\right): F\right]$ where F is the field of rational numbers.

9. What is the splitting field of the polynomial $x^2 - 3$ over $F\left(\sqrt{2}\right)$ where F is the field of rational numbers.

10. Give an example of a simple extension.

11. Give an example of a normal extension.

12. Find the Galois group of the polynominal $x^2 + 1$ over the field F of rational numbers.

13. Show that every abelian group is solvable.

14. Show that any two finite fields of p^n elements are isomorphic, where p is prime and n is a positive integer.

15. If S and T are nilpotent linear transformation which commute, prove that ST is also nilpotent.

16. Find the companion matrix of $(x+1)^3$

17. If $T \in A(V)$ is nilpotent, prove that $T = 0$.

18. If T is Skew-Hermitian, prove that all of its characteristic roots are pure imaginaries.

19. Prove that if T is Hermitian, prove (vT, v) is real for every $v \in V$.

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