DECEMBER 2015 **P/ID 37451/PMAA**

Time : Three hours Maximum : 100 marks

SECTION A — (10 × 2 = 20 marks)

Answer ALL questions.

1. Show that the conjugacy is an equivalence relation.

2. Define N(a) for any a in G. Prove that N(a) has atleast two elements.

3. Define an invariant under linear Transformation $T \in A(v)$.

4. Define companion matrix.

5. Define a simple extension.

6. Define an algebraic number.

7. Define the splitting field of the polynomial $f(x)$ over F.

8. Define a simple extension field.

9. Define an algebraic element over a field.

10. Define the basic Jordan block belonging to λ.

SECTION B — (5 × 6 = 30 marks)

Answer ALL questions.

11. (a) If $O(G) = p^2$ where p is a prime number then prove that G is abelian.

Or

(b) Prove that G is solvable if and only if $G^{(k)} = e$ for some integer k.

12. (a) If $T \in A(V)$ is nilpotent of index of nilpotence n_1, then prove that a basis of V can be found such that the matrix of T in this basis has the form $\begin{pmatrix} M_{n_1} & 0 & \cdots & 0 \\ 0 & M_{n_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & M_{n_r} \end{pmatrix}$.

where $n_1 \geq n_2 \geq \ldots \geq n_r$ and

$n_1 + n_2 + \ldots + n_r = \dim_f v$.

Or

(b) If the minimal polynomial of T_1 over F is $P_1(x)$ while that of T_2 is $P_2(x)$ then prove that the minimal polynomial for T over F is the least common multiple $P_1(x)$ and $P_2(x)$.

13. (a) If $T \in A(V)$ then prove that trace T is the sum of the characteristic root of T

Or

(b) If $T \in A(V)$ is such that $(uT, u) = 0$ for all $u \in$ V then prove that $T = 0$.

14. (a) For any $f(x), g(x) \in F[x]$ and any $d \in F$, then prove that

(i) $(f(x) + g(x))' = f'(x) + g'(x)$

(ii) $(\alpha f(x))' = \alpha f'(x)$

(iii) $(f(x)\,g(x))' = f'(x)g(x) + f(x)g'(x)$

Or

(b) Prove that K is a normal extension of F if and only if K is the spliting field of some polynomial over F.

15. (a) Prove that the general polynomial of degree $n \geq 5$ is not solvable by radicals.

Or

(b) If $a \in H$ then prove that $a^{-1} \in H$ if and only if $N(a) = 1$.

SECTION C — (5 × 10 = 50 marks)

Answer ALL questions.

16. (a) Prove that the number of conjugate classes in S_n is $P(n)$, the number of partition of n.

Or

(b) If $P(x) \in F[x]$ is solvable by radicals over F, then prove that the Galvis group over F of $P(x)$ is a solvable group.

17. (a) If $T \in A(V)$ has all its characteristic roots is F, then prove that there is a basis of V in which the matrix of T is triangular.

Or

(b) Prove that there exist a subspace W of V invariant under T, such that $V = V_1 \oplus W$.

18. (a) Prove that the number e is transcendental.

Or

(b) If $T \in A(V)$ then prove that given any $u \in V$ there exists an element $w \in V$ depending on v and T such that $(ut, v) = (u, w)$ for all $u \in V$. This element w is uniquely determined by v and T.

19. (a) If $P(x)$ is a polynomial in $F[x]$ of degree $n \geq 1$ and is irreducible over F then prove that there is an extension E of F such that $[E : F] = n$ in which $P(x)$ has a root.

Or

(b) Prove that the polynomial $f(x) \in F[x]$ has a multiple root if and only if $f(x)$ and $f'(x)$ have a nontrivial common factor.

20. (a) If F is a finite field and $\alpha \neq 0$, $\beta \neq 0$ are two elements of F, prove that there exits a and b such that $1 + \alpha a^2 + \beta b^2 = 0$.

Or

(b) Let G be a finite abelian group enjoying the property that the relation $x^n = e$ is satisfied by at most n element of G for every integer n. Then prove that G is a cyclic group.

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