(6 pages)

DECEMBER 2014 **P/ID 37451/PMAA**

Time : Three hours Maximum : 100 marks

SECTION A — (10 × 2 = 20 marks)

Answer ALL questions.

1. Show that conjugacy is an equivalence relation on a group G.

2. Show that every abelian group is solvable.

3. Define invariants of a linear transformation.

4. Define Jordan form of a linear transformation.

5. If $S \in A(V)$, show that $S = A + iB$ where both A and B are Hermitian.

6. Show that every finite extension is an algebraic extension.

7. Define a splitting field of a polynomial $f(x)$ over F.

8. Define a normal extension.

9. Write down the cyclotomic polynomial $\Phi_3(x)$.

10. Show that, for all $x, y \in Q$, the ring of real quaternions, $N(xy) = N(x)N(y)$.

SECTION B — (5 × 6 = 30 marks)

Answer ALL questions.

11. (a) Derive the class equation of a finite group.

Or

(b) Show that any two p-sylow subgroups of a group G are conjugate in G.

12. (a) If V is an n-dimensional vector space over F and if $T \in A(V)$ has all its characteristic roots in F, then prove that T satisfies a polynomial of degree n over F.

Or

(b) If a vector space V over F is cyclic relative to $T \in A(V)$ and if the minimal polynomial of T in $F[x]$ is $p(x)$, then prove that, for some basis of V, the matrix of T is $C(p(x))$.

13. (a) Show that all the characteristics roots of a Skew-Hermitian linear transformations are pure imaginaries.

Or

(b) Show that the elements in K which are algebraic over F Form a subfield of K.

14. (a) If $n \geq 1$, prove that the splitting field of $x^n - 1$ over the field of rational numbers is of degree $\Phi(n)$ where Φ is the Euler Φ -function.

Or

(b) If K is a finite extension of F, then prove that $G(K,F)$ is a finite group and its order $O(G(K,F))$ satisfies $O(G(K,F)) \leq [K:F]$.

15. (a) If G is a finite abelian group enjoying the property that the relation $x^n = e$ is satisfied by at most n elements of G, for every integer n, then prove that G is a cyclic group.

Or

(b) If A is a ring algebraic over a field F and A has no zero divisors prove that A is a division ring.

SECTION C — (5 × 10 = 50 marks)

Answer ALL questions.

16. (a) State and prove the third part of Sylow's theorem.

Or

(b) Show that any finitely generated module over a Euclidean ring is the direct sum of a finite number of cyclic sub modules.

17. (a) Show that there exists a subspace W of a vector space V, invariant under a nilpotent transformation T, such that $V = V_1 \oplus W$, where V_1 is the subspace of V spanned by $v,\ vT, ...vT^{n-1}$.

Or

(b) Show that S and T in $A_F(V)$ are similar if and only if they have the same elementary divisors.

18. (a) If F is of characteristic 0 and if a, b are algebraic over F, then show that there exists an element $c \in F(a, b)$ such that $F(a, b) = F(c)$.

Or

(b) Show that K is a normal extension of F if and only if K is the splitting field of some polynomial over F.

19. (a) Determine the rank and signature of the real quadratic form

$$x_1^2 + x_1x_2 + 2x_1x_3 + 2x_2^2 + 4x_2x_3 + 2x_3^2.$$

Or

(b) Show that $a \in K$ is algebraic over F if and only if F (a) is finite extension of F.

20. (a) Show that a finite division ring is necessarily a commutative field.

Or

(b) State and prove Frobenius theorem.
