(6 pages)

OCTOBER 2013 **P/ID 37451/PMAA**

Time : Three hours Maximum : 100 marks

SECTION A — (10 × 2 = 20 marks)

Answer ALL the questions.

Each question carries 2 marks.

1. State Cauchys theorem.

2. Define R-module.

3. Define elementary divisors

4. Define characteristic Polynomial.

5. Define the transpose of the matrix A.

6. Define unitary transformation.

7. Define simple extension

8. Define Galois group.

9. If the field F has p^m elements, prove that F is the splitting field of the Polynomial $x^{p^m} - x$.

10. State Left Division Algorithm

SECTION B — ($5 \times 6 = 30$ marks)

Answer ALL the questions.

Each questions carries 6 marks.

11. (a) State and prove second part of Sylow's theorem.

Or

(b) Let $G = S_n$, where $n \geq 5$. Prove that $G^{(k)}$ for k = 1,2….. contains every 3–cycle of S_n.

12. (a) Prove that two nilpotent transformations are similar if and only have the same invariants.

Or

(b) If $T \in A(V)$ is nilpotent, prove that $\alpha_0 + \alpha_1 T + ... + \alpha_m T^m$. Where $\alpha_i \in F$, is invertible if $\alpha_0 \neq 0$.

13. (a) For all $A, B \in F_n$, prove that $(AB)' = B'A'$.

Or

(b) Let N be a normal transformation and suppose that λ and μ are two distinct characteristic roots of N. If v, w are in V and are such that $vN = \lambda v, wN = \mu w$, prove that $(v, w) = 0$.

14. (a) If K is a field and if $\sigma_1, ...\sigma_n$ are distrinct automorphisms of K, prove that it is impossible to find elements $a_1.....a_n$, not all 0, in K such that $\sigma_1\sigma_1(u) + \alpha_2\sigma_2(u) + ... + \alpha_n\sigma_n(u) = 0$ for all $u \in K$.

Or

(b) Prove that K is a normal extension of F if and only if K is splitting field of some polynomial over F.

15. (a) Let F be a field with q elements and suppose that $F \subseteq K$, where K is also finite field. Prove that K has q^n elements, where $n = [k : F]$.

Or

(b) If $\alpha \in H$, prove that $\alpha^{-1} \in H$ if and only if $N(a) = 1$.

SECTION C — ($5 \times 10 = 50$ marks)

Answer ALL the questions.

Each question carries 10 marks.

16. (a) Prove that every finite abelian group is the direct product of cyclic groups.

Or

(b) If G is a finite group, prove that $C_a = 0(G)/0(N(a))$.

17. (a) If $T \in A(V)$ has all its characteristic roots in F, prove that there is a basis of V in which the matrix of T is triangular.

Or

(b) Let V and W be two vector spaces over F and suppose that ψ is a vector space isomorphism of V onto W. Suppose that $S \in A_F(V)$ and $T \in A_F(W)$ are such that for any $\upsilon \in V, (\upsilon S)\psi(\upsilon\psi)T$. Prove that S and T have the same elementary divisors.

18. (a) If $T \in A(V)$, prove that $T^* \in A(V)$. Further prove the following.

    (i) $(T^*) = T$

    (ii) $(S+T)^* = S^* + T^*$

    (iii) $(\lambda S)^* = \bar{\lambda} S^*$

    (iv) $(ST)^* = T^* S^*$

    For all $S, T \in A(V)$ and all $\lambda \in F$.

Or

(b) State and prove Wedderburn theorem.

19. (a) Prove that the polynomial $f(x) \in F[x]$ has a multiple root if $f(x)$ and $f'(x)$ have a nontrivial common factor.

Or

(b) If $p(x)$ is irreducible in $F[x]$ and if v is a root of $p(x)$, prove that $F(v)$ is isomorphic to $F'(w)$, where w is a root of $p'(t)$. Also prove that this isomorphism σ can so be chose that

    (i) $v\sigma = w$

    (ii) $\alpha\sigma = \alpha'$, for every $\alpha \in F$

20. (a) Let G be a finite abelian group enjoying the property that the relation $x^n = e$ is satisfied by atomost n elements of G, for every integer n. Prove that G is a cyclic group.

Or

(b) State and prove Frobenius theorem.

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