

INMO 2004 - Solutions

1. Consider a convex quadrilateral $ABCD$, in which K, L, M, N are the midpoints of the sides AB, BC, CD, DA respectively. Suppose
- BD bisects KM at Q ;
 - $QA = QB = QC = QD$; and
 - $LK/LM = CD/CB$.

Prove that $ABCD$ is a square.

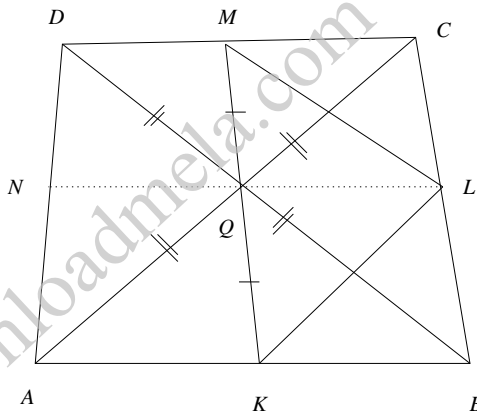
Solution:

Fig. 1.

Observe that $KLMN$ is a parallelogram, Q is the midpoint of MK and hence NL also passes through Q . Let T be the point of intersection of AC and BD ; and let S be the point of intersection of BD and MN .

Consider the triangle MNK . Note that SQ is parallel to NK and Q is the midpoint of MK . Hence S is the mid-point of MN . Since MN is parallel to AC , it follows that T is the mid-point of AC . Now Q is the circumcentre of $\triangle ABC$ and the median BT passes through Q . Here there are two possibilities:

- ABC is a right triangle with $\angle ABC = 90^\circ$ and $T = Q$; and
- $T \neq Q$ in which case BT is perpendicular to AC .

Suppose $\angle ABC = 90^\circ$ and $T = Q$. Observe that Q is the circumcentre of the triangle DCB and hence $\angle DCB = 90^\circ$. Similarly $\angle DAB = 90^\circ$. It follows that $\angle ADC = 90^\circ$. and $ABCD$ is a rectangle. This implies that $KLMN$ is a rhombus. Hence $LK/LM = 1$ and this gives $CD = CB$. Thus $ABCD$ is a square.

In the second case, observe that BD is perpendicular to AC , KL is parallel to AC and LM is parallel to BD . Hence it follows that ML is perpendicular to LK . Similar reasoning shows that $KLMN$ is a rectangle.

Using $LK/LM = CD/CB$, we get that CBD is similar to LMK . In particular, $\angle LMK = \angle CBD = \alpha$ say. Since LM is parallel to DB , we also get $\angle BQK = \alpha$. Since $KLMN$ is a cyclic quadrilateral we also get $\angle LNK = \angle LMK = \alpha$. Using the fact that BD is parallel to NK , we get $\angle LQB = \angle LNK = \alpha$. Since BD bisects $\angle CBA$, we also have $\angle KBQ = \alpha$. Thus

$$QK = KB = BL = LQ$$

and BL is parallel to QK . This gives QM is parallel to LC and

$$QM = QL = BL = LC$$

It follows that $QLCM$ is a parallelogram. But $\angle LCM = 90^\circ$. Hence $\angle MQL = 90^\circ$. This implies that $KLMN$ is a square. Also observe that $\angle LQK = 90^\circ$ and hence $\angle CBA = \angle LQK = 90^\circ$. This gives $\angle CDA = 90^\circ$ and hence $ABCD$ is a rectangle. Since $BA = BC$, it follows that $ABCD$ is a square.

2. Suppose p is a prime greater than 3. Find all pairs of integers (a, b) satisfying the equation

$$a^2 + 3ab + 2p(a + b) + p^2 = 0.$$

Solution: We write the equation in the form

$$a^2 + 2ap + p^2 + b(3a + 2p) = 0$$

Hence

$$b = \frac{-(a + p)^2}{3a + 2p}$$

is an integer. This shows that $3a + 2p$ divides $(a + p)^2$ and hence also divides $(3a + 3p)^2$. But, we have

$$(3a + 3p)^2 = (3a + 2p + p)^2 = (3a + 2p)^2 + 2p(3a + 2p) + p^2.$$

It follows that $3a + 2p$ divides p^2 . Since p is a prime, the only divisors of p^2 are $\pm 1, \pm p$ and $\pm p^2$. Since $p > 3$, we also have $p = 3k + 1$ or $3k + 2$.

Case 1: Suppose $p = 3k + 1$. Obviously $3a + 2p = 1$ is not possible. Infact, we get $1 = 3a + 2p = 3a + 2(3k + 1) \Rightarrow 3a + 6k = -1$ which is impossible. On the other hand $3a + 2p = -1$ gives $3a = -2p - 1 = -6k - 3 \Rightarrow a = -2k - 1$ and $a + p = -2k - 1 + 3k + 1 = k$.

Thus $b = \frac{-(a + p)^2}{(3k + 2p)} = k^2$. Thus $(a, b) = (-2k - 1, k^2)$ when $p = 3k + 1$. Similarly, $3a + 2p = p \Rightarrow 3a = -p$ which is not possible. Considering $3a + 2p = -p$, we get $3a = -3p$ or $a = -p \Rightarrow b = 0$. Hence $(a, b) = (-3k - 1, 0)$ where $p = 3k + 1$.

Let us consider $3a + 2p = p^2$. Hence $3a = p^2 - 2p = p(p - 2)$ and neither p nor $p - 2$ is divisible by 3. If $3a + 2p = -p^2$, then $3a = -p(p + 2) \Rightarrow a = -(3k + 1)(k + 1)$.

Hence $a + p = (3k + 1)(-k - 1 + 1) = -(3k + 1)k$. This gives $b = k^2$. Again $(a, b) = \left(-(k + 1)(3k + 1), k^2 \right)$ when $p = 3k + 1$.

Case 2: Suppose $p = 3k - 1$. If $3a + 2p = 1$, then $3a = -6k + 3$ or $a = -2k + 1$. We also get

$$b = \frac{-(a + p)^2}{1} = \frac{-(-2k + 1 + 3k - 1)^2}{1} = -k^2$$

and we get the solution $(a, b) = (-2k + 1, k^2)$. On the other hand $3a + 2p = -1$ does not have any solution integral solution for a . Similarly, there is no solution in the case $3a + 2p = p$. Taking $3a + 2p = -p$, we get $a = -p$ and hence $b = 0$. We get the solution $(a, b) = (-3k + 1, 0)$. If $3a + 2p = p^2$, then $3a = p(p - 2) = (3k - 1)(3k - 3)$ giving $a = (3k - 1)(k - 1)$ and hence $a + p = (3k - 1)(1 + k - 1) = k(3k - 1)$. This gives $b = -k^2$ and hence $(a, b) = (3k - 1, -k^2)$. Finally $3a + 2p = -p^2$ does not have any solution.

3. If α is a real root of the equation $x^5 - x^3 + x - 2 = 0$, prove that $[\alpha^6] = 3$. (For any real number a , we denote by $[a]$ the greatest integer not exceeding a .)

Solution: Suppose α is a real root of the given equation. Then

$$\alpha^5 - \alpha^3 + \alpha - 2 = 0. \quad \dots(1)$$

This gives $\alpha^5 - \alpha^3 + \alpha - 1 = 1$ and hence $(\alpha - 1)(\alpha^4 + \alpha^3 + 1) = 1$. Observe that $\alpha^4 + \alpha^3 + 1 \geq 2\alpha^2 + \alpha^3 = \alpha^2(\alpha + 2)$. If $-1 \leq \alpha < 0$, then $\alpha + 2 > 0$, giving $\alpha^2(\alpha + 2) > 0$ and hence $(\alpha - 1)(\alpha^4 + \alpha^3 + 1) < 0$. If $\alpha < -1$, then $\alpha^4 + \alpha^3 = \alpha^3(\alpha + 1) > 0$ and hence $\alpha^4 + \alpha^3 + 1 > 0$. This again gives $(\alpha - 1)(\alpha^4 + \alpha^3 + 1) < 0$.

The above reasoning shows that for $\alpha < 0$, we have $\alpha^5 - \alpha^3 + \alpha - 1 < 0$ and hence cannot be equal to 1. We conclude that a real root α of $x^5 - x^3 + x - 2 = 0$ is positive (obviously $\alpha \neq 0$).

Now using $\alpha^5 - \alpha^3 + \alpha - 2 = 0$, we get

$$\alpha^6 = \alpha^4 - \alpha^2 + 2\alpha$$

The statement $[\alpha^6] = 3$ is equivalent to $3 \leq \alpha^6 < 4$.

Consider $\alpha^4 - \alpha^2 + 2\alpha < 4$. Since $\alpha > 0$, this is equivalent to $\alpha^5 - \alpha^3 + 2\alpha^2 < 4\alpha$. Using the relation (1), we can write $2\alpha^2 - \alpha + 2 < 4\alpha$ or $2\alpha^2 - 5\alpha + 2 < 0$. Treating this as a quadratic, we get this is equivalent to $\frac{1}{2} < \alpha < 2$. Now observe that if $\alpha \geq 2$ then $1 = (\alpha - 1)(\alpha^4 + \alpha^3 + 1) \geq 25$ which is impossible. If $0 < \alpha \leq \frac{1}{2}$, then $1 = (\alpha - 1)(\alpha^4 + \alpha^3 + 1) < 0$ which again is impossible.

We conclude that $\frac{1}{2} < \alpha < 2$. Similarly $\alpha^4 - \alpha^2 + 2\alpha \geq 3$ is equivalent to $\alpha^5 - \alpha^3 + 2\alpha^2 - 3\alpha \geq 0$ which is equivalent to $2\alpha^2 - 4\alpha + 2 \geq 0$. But this is $2(\alpha - 1)^2 \geq 0$ which is valid. Hence $3 \leq \alpha^6 < 4$ and we get $[\alpha^6] = 3$.

4. Let R denote the circumradius of a triangle ABC ; a, b, c its sides BC, CA, AB ; and r_a, r_b, r_c its exradii opposite A, B, C . If $2R \leq r_a$, prove that

(i) $a > b$ and $a > c$;

(ii) $2R > r_b$ and $2R > r_c$.

Solution: We know that $2R = \frac{abc}{\Delta}$ and $r_a = \frac{\Delta}{s - a}$, where a, b, c are the sides of the triangle ABC ,

$s = \frac{a + b + c}{2}$ and Δ is the area of ABC . Thus the given condition $2R \leq r_a$ translates to

$$abc \leq \frac{2\Delta^2}{s - a}$$

Putting $s - a = p, s - b = q, s - c = r$, we get $a = q + r, b = r + p, c = p + q$ and the condition now is

$$p(p + q)(q + r)(r + p) \leq 2\Delta^2$$

But Heron's formula gives, $\Delta^2 = s(s - a)(s - b)(s - c) = pqr(p + q + r)$. We obtain $(p + q)(q + r)(r + p) \leq 2qr(p + q + r)$. Expanding and effecting some cancellations, we get

$$p^2(q + r) + p(q^2 + r^2) \leq qr(q + r). \quad (\star)$$

Suppose $a \leq b$. This implies that $q + r \leq r + p$ and hence $q \leq p$. This implies that $q^2r \leq p^2r$ and $qr^2 \leq pr^2$ giving $qr(q + r) \leq p^2r + pr^2 < p^2r + pr^2 + p^2q + pq^2 = p^2(q + r) + p(q^2 + r^2)$ which contradicts (\star) . Similarly, $a \leq c$ is also not possible. This proves (i).

Suppose $2R \leq r_b$. As above this takes the form

$$q^2(r + p) + q(r^2 + p^2) \leq pr(p + r). \quad (\star\star)$$

Since $a > b$ and $a > c$, we have $q > p, r > p$. Thus $q^2r > p^2r$ and $qr^2 > pr^2$. Hence

$$q^2(r + p) + q(r^2 + p^2) > q^2r + qr^2 > p^2r + pr^2 = pr(p + r)$$

which contradicts $(\star\star)$. Hence $2R > r_b$. Similarly, we can prove that $2R > r_c$. This proves (ii)

5. Let S denote the set of all 6-tuples (a, b, c, d, e, f) of positive integers such that $a^2 + b^2 + c^2 + d^2 + e^2 = f^2$. Consider the set

$$T = \{abcdef : (a, b, c, d, e, f) \in S\}.$$

Find the greatest common divisor of all the members of T .

Solution: We show that the required gcd is 24. Consider an element $(a, b, c, d, e, f) \in S$. We have

$$a^2 + b^2 + c^2 + d^2 + e^2 = f^2.$$

We first observe that not all a, b, c, d, e can be odd. Otherwise, we have $a^2 \equiv b^2 \equiv c^2 \equiv d^2 \equiv e^2 \equiv 1 \pmod{8}$ and hence $f^2 \equiv 5 \pmod{8}$, which is impossible because no square can be congruent to 5 modulo 8. Thus at least one of a, b, c, d, e is even.

Similarly if none of a, b, c, d, e is divisible by 3, then $a^2 \equiv b^2 \equiv c^2 \equiv d^2 \equiv e^2 \equiv 1 \pmod{3}$ and hence $f^2 \equiv 2 \pmod{3}$ which again is impossible because no square is congruent to 2 modulo 3. Thus 3 divides $abcdef$.

There are several possibilities for a, b, c, d, e .

Case 1: Suppose one of them is even and the other four are odd; say a is even, b, c, d, e are odd. Then $b^2 + c^2 + d^2 + e^2 \equiv 4 \pmod{8}$. If $a^2 \equiv 4 \pmod{8}$, then $f^2 \equiv 0 \pmod{8}$ and hence $2|a, 4|f$ giving $8|af$. If $a^2 \equiv 0 \pmod{8}$, then $f^2 \equiv 4 \pmod{8}$ which again gives that $4|a$ and $2|f$ so that $8|af$. It follows that $8|abcdef$ and hence $24|abcdef$.

Case 2: Suppose a, b are even and c, d, e are odd. Then $c^2 + d^2 + e^2 \equiv 3 \pmod{8}$. Since $a^2 + b^2 \equiv 0$ or $4 \pmod{8}$, it follows that $f^2 \equiv 3$ or $7 \pmod{8}$ which is impossible. Hence this case does not arise.

Case 3: If three of a, b, c, d, e are even and two odd, then $8|abcdef$ and hence $24|abcdef$.

Case 4: If four of a, b, c, d, e are even, then again $8|abcdef$ and $24|abcdef$. Here again for any six tuple (a, b, c, d, e, f) in S , we observe that $24|abcdef$. Since

$$1^2 + 1^2 + 1^2 + 2^2 + 3^2 = 4^2.$$

We see that $(1, 1, 1, 2, 3, 4) \in S$ and hence $24 \in T$. Thus 24 is the gcd of T .

6. Prove that the number of 5-tuples of positive integers (a, b, c, d, e) satisfying the equation

$$abcde = 5(bcde + acde + abde + abce + abcd)$$

is an **odd** integer.

Solution: We write the equation in the form:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} = \frac{1}{5}.$$

The number of five tuple (a, b, c, d, e) which satisfy the given relation and for which $a \neq b$ is even, because for if (a, b, c, d, e) is a solution, then so is (b, a, c, d, e) which is distinct from (a, b, c, d, e) . Similarly the number of five tuples which satisfy the equation and for which $c \neq d$ is also even. Hence it suffices to count only those five tuples (a, b, c, d, e) for which $a = b, c = d$. Thus the equation reduces to

$$\frac{2}{a} + \frac{2}{c} + \frac{1}{e} = \frac{1}{5}.$$

Here again the tuple (a, a, c, c, e) for which $a \neq c$ is even because we can associate different solution (c, c, a, a, e) to this five tuple. Thus it suffices to consider the equation

$$\frac{4}{a} + \frac{1}{e} = \frac{1}{5},$$

and show that the number of pairs (a, e) satisfying this equation is odd.

This reduces to

$$ae = 20e + 5a$$

or

$$(a - 20)(e - 5) = 100.$$

But observe that

$$\begin{aligned} 100 &= 1 \times 100 = 2 \times 50 = 4 \times 25 = 5 \times 20 \\ &= 10 \times 10 = 20 \times 5 = 25 \times 4 = 50 \times 2 = 100 \times 1. \end{aligned}$$

Note that no factorisation of 100 as product of two negative numbers yield a positive tuple (a, e) . Hence we get these 9 solutions. This proves that the total number of five tuples (a, b, c, d, e) satisfying the given equation is odd.



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